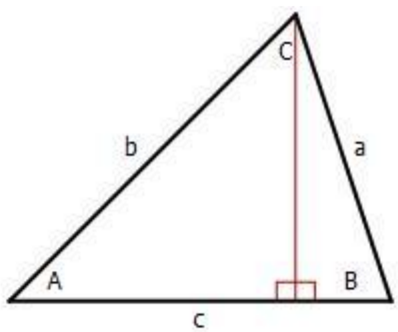
White Paper
Topic: The Law of Sines

What's Inside:

- Origin of the formula for the Law of Sines
- How to Use the Law of Sines to find missing parts of a triangle
- Dealing with "the ambiguous case" of the Law of Sines

For an oblique (that is, a non-right) triangle, it can be difficult to understand how the sides and angles relate. After all, since it is not a right triangle, there are no simple formulas like the Pythagorean Theorem to relate the side, no simple rules like "SOH-CAH-TOA" to relate the angles to the sides. Also, knowing the measures of any one of the acute angles of an oblique triangle does not guarantee that we will know that we will know the measure of any other angle of the triangle, as it would if we were dealing with a right triangle.

However, there is a way to deal with these problems that are associated with oblique triangles. If we draw an altitude from any vertex to the line containing the opposite side, we can make some interesting conclusions regarding that altitude. Let's try this with a generic triangle ABC:

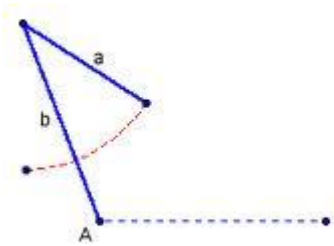


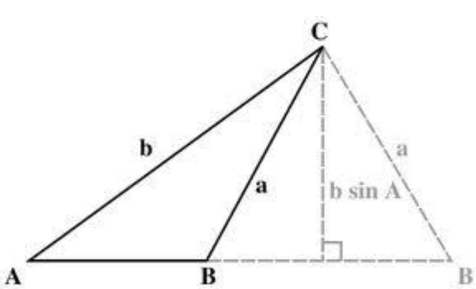
I have drawn an altitude from vertex C to $\overline{AB}$. See how this creates two right triangles? Let's say that we wanted to find the length of the altitude (shown in red—I'll call it "h"). Using the larger right triangle (on the left), we can state that $\sin A = \frac{h}{b}$. Similarly, using the smaller right triangle (on the right), we can state that $\sin B = \frac{h}{a}$. Solving each equation for h, we can conclude $h = bsin\,A$ and $h = a\sin B$. Since the two quantities are equal to h, they must be equal to each other. Thus, $a \sin B = b \sin A$, and finally, $\frac{\sin A}{a} = \frac{sinB}{b}$. We can make a similar argument for side c and angle C as well (by drawing a different altitude). Thus we can extend this proportion to $\frac{\sin A}{a} = \frac{sinB}{b} = \frac{sinC}{c}$. This extended proportion is known as **The Law of Sines**.

So, when is it appropriate to use the Law of Sines? Depending on the given information that you have, you must be able to directly plug in three out of any four quantities in a proportion in order for it to work (I like to call it having "a matching pair" of a side and an angle PLUS one other piece of information. Sometimes the given information allows this, and sometimes it must be inferred. For example, given angle A and angle B, and side c, we can get our matching pair from side c and angle C (C is not directly given but can be found by angle sums).

I'm going to give you a uniform approach to finding all of the missing information, but first I want to tell you about a special situation that sometimes occurs with the Law of Sines. Given the SSA condition we can find ourselves in a case where 0, 1, or 2 triangles may be constructed to match the given information in the problem. Perhaps this diagram will be helpful in understanding how this is possible.

In the figure at the right, we can see that side a, side b, and angle A are not sufficient to make a triangle. This is because the length of a does not meet the unstated minimum requirement that it be at least as long as the altitude. So if you are given a situation like this, you are going to find that no solutions are possible for this type of situation.





In the figure at the left, you can see that angle A, side a, and side b are sufficient to have two distinct triangles which satisfy the given conditions. Note that although it is not explicitly stated in the diagram, the two versions of angle B are supplements of each other. (more on this in a minute!)

All other cases have exactly one solution, and I'd like to walk you through how to approach problems that can be solved using the Law of Sines. It is not even important that you know how many solutions exist. No knowledge of the situation is necessary, but of course, the more you know in advance, the better off you will be in terms of total understanding.

Here's an example to show you how to deal with the situations that arise.

Example 1. Solve the triangle with $a = 20, b = 30, A = 80^o$.

We have a "matching pair", so we can use the Law of Sines to solve.

$$\frac{\sin 80^o}{20} = \frac{\sin B}{30}$$

$$30 \sin 80^o = 20 \sin B$$

$$\sin B = \frac{30 \sin 80^o}{20} \approx 1.4772$$

There is no solution, since the range for the sine function must be contained within $[-1, 1]$.

Example 2. Solve the triangle with $D = 30^o, d = 15, e = 12$.

Again, we have a matching pair. The Law of Sines is appropriate for solving the problem.

$$\frac{\sin 30^o}{15} = \frac{\sin E}{12}$$

$$15 \sin E = 12 \sin 30^o$$

$$\sin E = \frac{12 \sin 30^o}{15} = 0.4$$

$$E = \sin^{-1} 0.4 \approx 23.6^o.$$

Here is where the analysis begins. Remember before when I said that the supplement to the angle was important? What I want you to do when you find the answer is to determine the value of the supplement to the angle you just got. I'm going to label it E', as in a variation of E.

$$E' = 180^o - 23.6^o = 156.4^o$$

Now ask yourself if that value of E' plus the original angle is less than 180^o. If it is, then a second triangle exists which satisfies the original conditions. In our case, $156.4^o + 30^o = 186.4^o$. Since this value exceeds 180^o, then there is no second triangle. Thus, all you have to do is find the remaining side of the original triangle you were working on. So we have . . .

$D = 30^o, d = 15, e = 12$ (given)
$E = 23.6^o$
Thus, $F = 180^o - 30^o - 23.6^o = 126.4^o$.

So, all we have to do is find f.

$$\frac{\sin 30^o}{15} = \frac{\sin 126.4^o}{f}$$

Thus,

$$e = \frac{15 \sin 126.4^o}{\sin 30^o} \approx 24.1$$

And the problem is complete.

Let’s try one more where the problem has two triangles which satisfy the given conditions.

Example. Solve the triangle ABC where $a = 12, b = 31, A = 20.5^o$.

We begin with our matching pair.

$$\frac{\sin 20.5^o}{12} = \frac{\sin B}{31}$$

$$12 \sin B = 31 \sin 20.5^o$$

$$\sin B = \frac{31 \sin 20.5^o}{12} \approx 0.9047$$

$$B \approx 64.8^o$$

Now, immediately examine the supplement of the angle you just found!

$$B' \approx 115.2^o$$

Note that this angle, plus the original angle, 20.5^o is less than 180^o! This means that there MUST be a second triangle. We will solve them separately.

$a = 12, b = 31, A = 20.5^o$. (given) $B \approx 64.8^o$ (Law of Sines) $C \approx 115.2^o$ (angle addition) Now just find c! $\frac{\sin 20.5^o}{12} = \frac{\sin 115.2^o}{c}$ $c\sin 20.5^o = 12 \sin 115.2^o$ $c = \frac{12 \sin 115.2^o}{\sin 20.5^o} \approx 31.7$	$a = 12, b = 31, A = 20.5^o$. (given) $B' \approx 115.2^o$ (Law of Sines) $C' \approx 44.3^o$ (angle addition) Now just find c'! $\frac{\sin 20.5^o}{12} = \frac{\sin 44.3^o}{c'}$ $c\sin 20.5^o = 12 \sin 44.3^o$ $c' = \frac{12 \sin 44.3^o}{\sin 20.5^o} \approx 23.9$

A note about rounding.

When doing these types of problems, I would advise that you either do all the work in one step or use the memory feature of your calculator to retain the previous answer. If you use a rounded form of a number in one step of a problem and use that to make further calculations, you may end up causing rounding errors (that although small, may be significant).

You may know that on the TI-8X calculators, hitting [2nd] [(-)] will bring up your preserved, previous answer. It will not round it off in the process. If you require multiple steps to make a calculation (nothing wrong with that!), please use the previous answer as your next step in your work.